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A Simple Graphical Representation of Fourier-Based Imaging Methods

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In all Fourier-based NMR imaging methods the envelope of the normalized complex FID signal in the rotating frame can be written (*1*)

$$S(t) = \int \rho(\mathbf{r}) \exp\left(i\gamma\mathbf{r}\cdot\int_0^t \mathbf{G}(t')dt'\right)d\mathbf{r} \qquad [1]$$

where $\mathbf{G}(t)$ is the gradient of the magnetic flux density

$$\mathbf{G} = \nabla B_z = (\nabla\mathbf{B})\cdot\mathbf{e}_z, \qquad [2]$$

t is the time after the exciting 90° pulse, γ is the magnetogyric factor, and $\rho(\mathbf{r})$ the effective spin density distribution at the point $\mathbf{r}$. For times not much less than T_2 the right hand side of [1] should be multiplied by $\exp(-t/T_2)$. To fix the ideas we shall further assume that a thin layer of the sample at right angles to the z axis has been selectively excited and that $\mathbf{r}$ is a vector in the xy plane. However, the argument will be equally valid in the one- or three-dimensional cases.

Now, the Fourier transform of the spin density is defined as

$$\hat{\rho}(\mathbf{k}) = \int \rho(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} \qquad [3]$$

which implies that

$$S(t) = \hat{\rho}(\mathbf{k}(t)) \qquad [4]$$

with $\mathbf{k}(t)$ defined by

$$\mathbf{k}(t) = \gamma\int_0^t \mathbf{G}(t')dt'. \qquad [5]$$

This is trivial in the case of a constant gradient. However, in the general case it allows the following simple visual interpretation. With increasing t the function $\mathbf{k}(t)$ describes a trajectory scanning the $\mathbf{k}$ plane, the value of the Fourier transform $\hat{\rho}(\mathbf{k})$ of the spin density distribution being given by the value $S(t)$ at each point of the trajectory.

In addition, we observe that according to the Riemann–Lebesgue lemma (*2*) the value of $\rho(\mathbf{k})$ will tend to zero when $|\mathbf{k}|$ tends to infinity. For all normal spin density functions this decay with growing $|\mathbf{k}(t)|$ will be quite rapid. However, as long as the spins have not been irreversibly defocused by the spin–spin relaxation the signal can always be *recalled* by letting the trajectory return to the region of small $|\mathbf{k}|$ values,

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thus giving rise to an "echo." Since the time interval between the exciting 90° pulses is determined by the relaxation time, failure to recall the signal as often as possible prior to its final decay would thus imply sacrificing a potential source of information.

One way of recalling the signal, i.e., of producing an echo, is to switch the dominant component of the gradient as it is done in the echo planar method (*1, 6*). Another one is the utilization of 180° pulses although these will usually have to be accompanied by switching one component of the gradient (usually the smaller one) in order to avoid a mere repetition of one and the same echo (*6*). A 180° pulse will obviously give rise to complex conjugation of the FID signal, i.e., to a change of the sign of **k** and to a complex phase factor which is equal to -1 if the phase of the pulse carrier wave is chosen in the usual way (with the field along the x' axis in the rotating coordinate system). This means that $\rho(\mathbf{k}(t))$ will be equal to $-S(t)$ after an odd number of 180° pulses.

In the case of periodically modulated field gradients a Fourier series expansion in the time domain of the function $\exp(i\mathbf{k}(t)\cdot\mathbf{r})$ may be the best way of effecting the image reconstruction (*4, 5*). However, the picture of the trajectory in the **k** plane may provide a more direct and visual basis for the comparison of different methods and for the construction of new variants. Some of the current NMR imaging methods will now be examined from this unified viewpoint. Figure 1 shows the corresponding trajectories in the **k** plane. For the sake of convenience we shall limit ourselves to two-dimensional cases.

In the projection reconstruction method (*1, 6*), Fig. 1a, a constant gradient is applied in the direction θ. No attempt is made to recall the signal and the whole sample must be allowed to relax before another projection is performed. Figure 1b illustrates the line scan method (*1, 7*) where a single projection is made of a selectively excited narrow strip of the sample. In the Fourier imaging method (*1, 8*), Fig. 1c, a gradient in the y direction is imposed during the first part t_y of the free induction decay and the signal is then observed at a constant gradient in the x direction. The experiment is repeated with different values of $G_y t_y$. In the echo planar imaging (EPI)

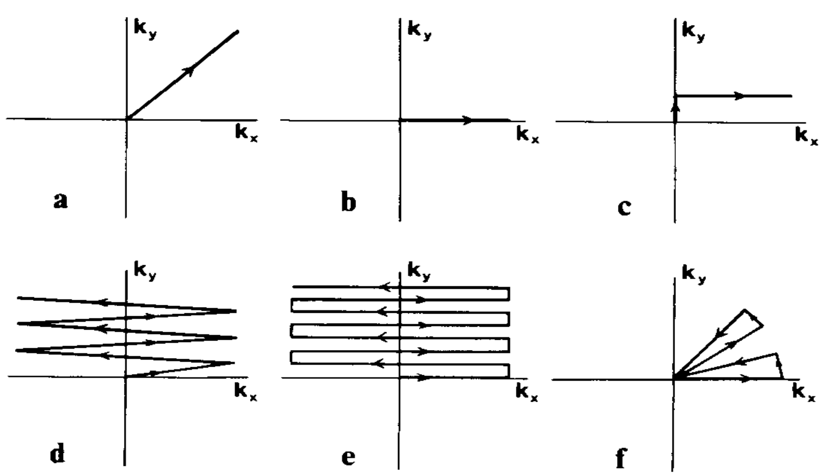

FIG. 1. Scanning patterns in the **k** plane: (a) projection reconstruction method; (b) line scan method; (c) Fourier imaging; (d) echo planar imaging method; (e) modified EPI method; (f) modified projection reconstruction method.

method (*1, 3–5*), Fig. 1d, the possibility of recalling the signal is fully utilized and the **k** plane is scanned along a zigzag curve. The gradient switching scheme that accomplishes this effect is shown in Fig. 2a. An echo appears each time when k_x attains a low value. It should be pointed out that the values of $\hat{\rho}$ for negative k_y values are easily obtained from the relation

$$\hat{\rho}(-\mathbf{k}) = \hat{\rho}^*(\mathbf{k}) \quad [6]$$

which is a consequence of $\rho(\mathbf{r})$ being real.

In the EPI method, k_y will be proportional to the time

$$k_y = \gamma G_y t \quad [7]$$

since G_y is constant during the whole experiment. This may be utilized to investigate the effect of the spin–spin relaxation factor $\exp(-t/T_2)$, which was omitted in Eq. [1]. Expressing t in k_y, this factor becomes $\exp(-k_y/\gamma G_y T_2)$, i.e.,

$$S(t) = \exp(-k_y/\gamma G_y T_2)\hat{\rho}(\mathbf{k}(t)). \quad [8]$$

The spin density distribution calculated from the experiment will thus be equal to the convolution of the true spin density function $\rho(\mathbf{r})$ with the function

$$\frac{2\gamma G_y T_2}{1 + (\gamma G_y y T_2)^2} \quad [9]$$

which is the Fourier transform of $\exp(-|k_y/\gamma G_y T_2|)$. This complication which also appears in Tropper's work (*4*) may be corrected for by a simple deconvolution procedure.

One possible disadvantage of the EPI method is that the values of $\rho(\mathbf{k})$ are given on a zigzag trajectory in the **k** plane rather than on the points of a regular rectangular lattice which might perhaps be more convenient from a computational point of view. To illustrate the capabilities of the method of representation described above, a modified scanning pattern has been devised, Fig. 1e. The corresponding gradient switching scheme is shown in Fig. 2b. In this case the values of $\hat{\rho}(\mathbf{k})$ will be given on a rectangular lattice of points and $\hat{\rho}(\mathbf{k})$ is easily obtained by an ordinary discrete Fourier transfor-

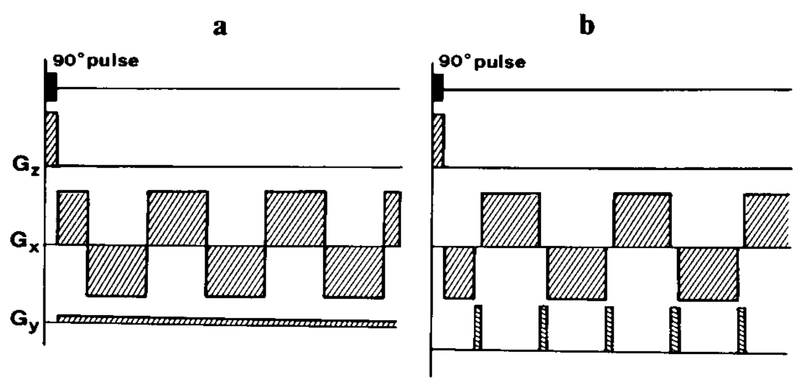

FIG. 2. Gradient switching scheme in (a) the EPI method, and (b) the modified EPI method.

mation. More generally, it is easy to construct a gradient time dependence corresponding to any arbitrary scanning path $\mathbf{k}(t)$ in the $\mathbf{k}$ plane, since we have

$$G(t) = (1/\gamma)(d\mathbf{k}(t)/dt). \qquad [10]$$

As an example, let us take a spiral scanning scheme

$$\begin{cases} k_x = at \cos bt \\ k_y = at \sin bt \end{cases} \qquad [11]$$

which is obtained if the gradient has the following time dependence:

$$\begin{cases} G_x = a \cos bt - abt \sin bt \\ G_y = a \sin bt + abt \cos bt. \end{cases} \qquad [12]$$

A scanning pattern with trajectories in the form of concentric circles might be preferable from a computational point of view. However, it is not the purpose of this communication to enter into a detailed discussion of the advantages or disadvantages of different experimental techniques. It would, of course, also be possible to improve the projection reconstruction method by recalling the signal and performing several successive projections after each excitation, Fig. 1f.

Since the relaxation causes a decrease of the signal amplitude with time it might be advantageous to reverse the scanning pattern by starting the scanning in the exterior regions of the $\mathbf{k}$ plane since the signal has its lowest amplitude in these parts. This would be equivalent to a kind of apodization or filtering.

Naturally, the whole arsenal of digital image processing developed in optics is directly applicable to NMR imaging. In fact, this kind of processing is greatly facilitated by the fact that the primary data are obtained in the form of the Fourier transform of the spin density distribution which means that no extra computational work is required for the filtering. In particular, the close analogy between $|\mathbf{k}|_{\max}$ and the numerical aperture in optics should be pointed out.

As a final application of this technique of visualization it is amusing to see how additional information could be obtained even in the one-dimensional line scan method. The most obvious extension of this method consists of recalling the signal one or several times by simply changing the sign of the gradient G_x. It is, however, also possible to obtain some information about the spin density variation in a direction perpendicular to the direction of the strip. If the strip is limited by $y_0 - \Delta y/2$ and $y_0 + \Delta y/2$ where Δy is a small quantity let us assume that the spin density function can be approximated by a truncated Taylor expansion

$$\rho(x, y) = \rho(x, y_0) + (y - y_0)\rho_y'(x, y_0) \equiv f(x) + (y - y_0)g(x). \qquad [13]$$

We then obtain

$$S(t) = \int_{-\infty}^{+\infty} \int_{y_0-\Delta y/2}^{y_0+\Delta y/2} [f(x) + (y - y_0)g(x)] \exp[i(k_x x + k_y y)]dxdy$$

$$= 2 \exp(ik_y y_0) \sin (k_y \Delta y/2)(1/k_y)\hat{f}(k_x) + (i\Delta y/k_y)[\cos (k_y \Delta y/2) - 2 \sin (k_y \Delta y/2)/k_y \Delta y] \exp(ik_y y_0)\hat{g}(k_x). \qquad [14]$$

The first measurement is made with $G_y = 0$ and at a constant G_x so that $k_x = \gamma G_x t$. Then

$$S(t) = (\Delta y)\hat{f}(k_x). \qquad [15]$$

Then k_y is increased from 0 to $2\pi/\Delta y$ by applying a gradient in the y direction of strength $G_y = 2\pi/\gamma\Delta y\Delta t$ for a time Δt, after which the sign of G_x is changed and the following signal is obtained:

$$S(t) = (\Delta y)^2\hat{g}(k_x)/2 \qquad [16]$$

where we have discarded the irrelevant phase factor $-i\exp(2\pi i y_0)$. A mathematically more exact interpretation of this experiment would be in terms of the Fourier components of $\rho(x, y)$ as a function of y.

The graphical method might even be extended to include approximately the case of a moving spin distribution as will be briefly outlined below. If $\rho(\mathbf{r}, \mathbf{v})$ denotes the probability density of spins at the point $\mathbf{r}$ moving with the velocity $\mathbf{v}$, the FID signal will be approximated by

$$S(t) = \iint \rho(\mathbf{r}, \mathbf{v}) \exp\left(i\gamma\mathbf{r}\cdot\int_0^t \mathbf{G}(t')dt'\right)\exp\left(i\gamma\mathbf{v}\cdot\int_0^t t'\mathbf{G}(t')dt'\right)d\mathbf{r}d\mathbf{v} \qquad [17]$$

at short times t. If we introduce $\mathbf{k}(t)$ according to Eq. [5] and

$$\mathbf{l}(t) = \int_0^t t'\mathbf{G}(t')dt' \qquad [18]$$

the expression for $S(t)$ may be written

$$S(t) = \iint \rho(\mathbf{r}, \mathbf{v})e^{i(\mathbf{k}\cdot\mathbf{r}+\mathbf{l}\cdot\mathbf{v})}d\mathbf{r}d\mathbf{v}. \qquad [19]$$

The expression can be further simplified if we introduce the direct sums

$$\mathbf{R} = \mathbf{r} \oplus \mathbf{v}$$
$$\mathbf{K} = \mathbf{k} \oplus \mathbf{l}. \qquad [20]$$

Then

$$S(t) = \int \rho(\mathbf{R})e^{i\mathbf{K}\cdot\mathbf{R}}d\mathbf{R} \qquad [21]$$

where $d\mathbf{R}$ signifies $d\mathbf{r}d\mathbf{v}$. The vector $\mathbf{K}(t)$ will thus trace a trajectory in the phase space spanned by the (four-dimensional) vectors $\mathbf{R}$ and the value of the Fourier transform of $\rho(\mathbf{R})$ is equal to $S(t)$ at each point of the trajectory. The function $\rho(\mathbf{r}, \mathbf{v})$ is thus in principle obtainable by inversion of the Fourier transform of Eq. [21].

Unfortunately, the trajectory $\mathbf{K}(t)$ cannot, as a rule, be chosen at will because of the functional dependence

$$d\mathbf{l}/dt = t(d\mathbf{k}/dt). \qquad [22]$$

To some extent this problem may be overcome by varying the magnitude of the gradient field. Thus, in the one-dimensional case (line scan method) we have, for a constant G_x,

$$k_x = G_x t$$
$$l_x = G_x t^2/2 = k_x^2/2G_x. \qquad [23]$$

This represents a family of parabolas in the $k_x l_x$ plane with G_x as a parameter.

The rather loosely outlined applications suggested above should not be taken too seriously. We believe, however, that the suggested method of presentation, consistently applied, may prove useful for clarifying and comparing different methods of NMR imaging. This method of presentation appears completely self-evident once it has been understood, and it is believed that many unnecessary mathematical complications such as the use of the so-called projection functions may be avoided in this way.

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