Sum of Two Sinusoids

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Several months ago I reviewed the manuscript of a book being considered by the IEEE Press for possible publication. In that manuscript the author presented the following equation:

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2} \cdot \cos[\omega t + \pi / 4]. \qquad (1)$$

Being unfamiliar with Eq. (1), and being my paranoid self, I wondered if that equation is correct. Not finding a stock trigonometric identity in my math reference book to verify Eq. (1), I modeled both sides of the equation using software. Sure enough, Eq. (1) is *not* correct. So then I wondered, "Humm ... OK, well then just what are the correct equations for a single sinusoid equivalent of the sum of equal-frequency sine and cosine functions? This can't be too difficult." As so often happens to me, the answers to that simple question are much more involved than I first thought.

Why Care About The Sum of Two Sinusoidal Functions

We frequently encounter the notion of the sum of two equal-frequency (real-valued) sinusoidal functions in the literature and applications of DSP. For example, some authors discuss this topic as a prelude to introducing the concept of negative frequency [1], or in their discussions of eigenfunctions [2,3]. Also, the sum of two equal-frequency sinusoids can be used to generate information-carrying signals in many digital communications systems, as well as explain the effects of what is called multipath fading of radio signals [4].

The Sum of Two Real Sinusoidal Functions

As it turns out, as you might expect, the sum of two equal-frequency real sinusoids is itself a single real sinusoid. However, the exact equations for all the various forms of that single equivalent sinusoid are difficult to find in the signal processing literature. Here we provide those equations:

- Table 1 gives the sum of two arbitrary cosine functions.
- Table 2 gives the sum of two arbitrary sine functions.
- Table 3 gives the sum of an arbitrary cosine and an arbitrary sine function.

In those tables, variables A and B are scalar constants, frequency ω is in radians/second, and variables α and β are phase angles measured in radians. The various forms of the sum of two real sinusoids are in the leftmost table columns. The single-sinusoid equivalents are in the rightmost columns. (Their derivations are provided at the end of this material.) As an example, the sixth row of Table 3 tells us that the correct form for the above incorrect Eq. (1) is:

$$A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2} \cdot \cos[\omega t - \tan^{-1}(B / A)]. \qquad (2)$$

Table 1: Sum of two real cosine waves

Sum of two cosine waves	Single sinusoid equivalent [1]
$A\cos(\omega t + \alpha) + B\cos(\omega t + \beta)$	$= \sqrt{[A\cos(\alpha) + B\cos(\beta)]^2 + [A\sin(\alpha) + B\sin(\beta)]^2} \bullet \cos\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha) + B\sin(\beta)}{A\cos(\alpha) + B\cos(\beta)}\right]\right\}$
$A\cos(\omega t + \alpha) + B\cos(\omega t + \alpha)$	$= \sqrt{A^2 + B^2 + 2AB} \bullet \cos(\omega t + \alpha)$
$A\cos(\omega t + \alpha) + B\cos(\omega t)$	$= \sqrt{A^2 + B^2 + 2AB\cos(\alpha)} \bullet \cos\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha)}{A\cos(\alpha) + B}\right]\right\}$
$A\cos(\omega t) + B\cos(\omega t)$	$= (A + B)\cdot\cos(\omega t)$
$A\cos(\omega t + \alpha) + A\cos(\omega t + \beta)$	$= \sqrt{2A^2[1 + \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)]} \bullet \cos\left\{\omega t + \tan^{-1}\left[\frac{\sin(\alpha) + \sin(\beta)}{\cos(\alpha) + \cos(\beta)}\right]\right\}$
$A\cos(\omega t + \alpha) + A\cos(\omega t + \alpha)$	$= 2A\cdot\cos(\omega t + \alpha)$
$A\cos(\omega t + \alpha) + A\cos(\omega t)$	$= 2A\cdot\cos(\alpha/2)\cdot\cos(\omega t + \alpha/2)$
$A\cos(\omega t) + A\cos(\omega t)$	$= 2A\cdot\cos(\omega t)$

1. The "•" symbol, needed for text wraparound reasons, means multiply.

Table 2: Sum of two real sinewaves

Sum of two sinewaves	Single sinusoid equivalent
$A\sin(\omega t + \alpha) + B\sin(\omega t + \beta)$	$= \sqrt{[A\cos(\alpha) + B\cos(\beta)]^2 + [A\sin(\alpha) + B\sin(\beta)]^2} \bullet \sin\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha) + B\sin(\beta)}{A\cos(\alpha) + B\cos(\beta)}\right]\right\}$
$A\sin(\omega t + \alpha) + B\sin(\omega t + \alpha)$	$= \sqrt{A^2 + B^2 + 2AB} \bullet \sin(\omega t + \alpha)$
$A\sin(\omega t + \alpha) + B\sin(\omega t)$	$= \sqrt{A^2 + B^2 + 2AB\cos(\alpha)} \bullet \sin\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha)}{A\cos(\alpha) + B}\right]\right\}$
$A\sin(\omega t) + B\sin(\omega t)$	$= (A + B)\cdot\sin(\omega t)$
$A\sin(\omega t + \alpha) + A\sin(\omega t + \beta)$	$= \sqrt{2A^2 [1 + \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)]} \bullet \sin\left\{\omega t + \tan^{-1}\left[\frac{\sin(\alpha) + \sin(\beta)}{\cos(\alpha) + \cos(\beta)}\right]\right\}$
$A\sin(\omega t + \alpha) + A\sin(\omega t + \alpha)$	$= 2A\cdot\sin(\omega t + \alpha)$
$A\sin(\omega t + \alpha) + A\sin(\omega t)$	$= 2A\cdot\cos(\alpha/2)\cdot\sin(\omega t + \alpha/2)$
$A\sin(\omega t) + A\sin(\omega t)$	$= 2A\cdot\sin(\omega t)$

Table 3: Sum of a real cosine wave and a real sinewave

Sum of a cosine wave and a sinewave	Single sinusoid equivalent
$A\cos(\omega t+\alpha)+B\sin(\omega t+\beta)$	$=\sqrt{[A\cos(\alpha)+B\sin(\beta)]^2+[A\sin(\alpha)-B\cos(\beta)]^2}$ $\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{A\sin(\alpha)-B\cos(\beta)}{A\cos(\alpha)+B\sin(\beta)}\right]\right\}$
$A\cos(\omega t+\alpha)+B\sin(\omega t+\alpha)$	$=\sqrt{A^2+B^2}\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{A\sin(\alpha)-B\cos(\alpha)}{A\cos(\alpha)+B\sin(\alpha)}\right]\right\}$
$A\cos(\omega t+\alpha)+B\sin(\omega t)$	$=\sqrt{A^2+B^2-2AB\sin(\alpha)}$ $\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{A\sin(\alpha)-B}{A\cos(\alpha)}\right]\right\}$
$A\cos(\omega t)+B\sin(\omega t+\beta)$	$=\sqrt{A^2+B^2+2AB\sin(\beta)}$ $\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{-B\cos(\beta)}{A+B\sin(\beta)}\right]\right\}$
$A\cos(\omega t)+B\sin(\omega t)$	$=\sqrt{A^2+B^2}\bullet\cos[\omega t-\tan^{-1}(B/A)]$
$A\cos(\omega t+\alpha)+A\sin(\omega t+\beta)$	$=\sqrt{2A^2[1+\cos(\alpha)\sin(\beta)-\sin(\alpha)\cos(\beta)]}$ $\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{\sin(\alpha)-\cos(\beta)}{\cos(\alpha)+\sin(\beta)}\right]\right\}$
$A\cos(\omega t+\alpha)+A\sin(\omega t+\alpha)$	$=\sqrt{2A^2}\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{\sin(\alpha)-\cos(\alpha)}{\cos(\alpha)+\sin(\alpha)}\right]\right\}$
$A\cos(\omega t+\alpha)+A\sin(\omega t)$	$=\sqrt{2A^2[1-\sin(\alpha)]}$ $\bullet\cos\left\{\omega t+\tan^{-1}\left[\frac{A[\sin(\alpha)-1]}{A\cos(\alpha)}\right]\right\}$
$A\cos(\omega t)+A\sin(\omega t+\beta)$	$=2A\cdot\cos(\beta/2-\pi/4)\cdot\cos(\omega t+\beta/2-\pi/4)$
$A\cos(\omega t)+A\sin(\omega t)$	$=\sqrt{2}\cdot A\cdot\cos(\omega t-\pi/4)$

NOTE: Several months after I created the equations in the above tables I ran across somewhat similar material on the Internet written by the prolific Julius O. Smith III. In that material Prof. Smith presents equations for the general case of summing $N \geq 2$ arbitrary cosine functions of the same frequency. That material can be found at the following web page:
http://ccrma.stanford.edu/~jos/filters/Sum_Sinusoids_Same_Frequency.html

Derivation Methods

Deriving the closed-form expressions for the sum of two equal-frequency sinusoidal functions is most easily accomplished by first finding the expression for the sum of two arbitrary equal-frequency complex exponentials. So that's where I started.

The Sum of Two Complex Exponentials

First we identify a general complex exponential as:

$$Ae^{j(\omega t+\alpha)} = Ae^{j\alpha}e^{j\omega t} \tag{3}$$

where the exponential's magnitude is the constant scalar *A*. Frequency ω is in radians/second, and α is a constant phase shift measured in radians. To add two general complex exponentials of the same frequency, we convert them to rectangular form and perform the addition as:

$$Ae^{j(\omega t+\alpha)} + Be^{j(\omega t+\beta)} = e^{j\omega t}\left(Ae^{j\alpha} + Be^{j\beta}\right)$$

$$= e^{j\omega t}\left[A\cos(\alpha) + jA\sin(\alpha) + B\cos(\beta) + jB\sin(\beta)\right]$$

$$= e^{j\omega t}\left[A\cos(\alpha) + B\cos(\beta) + j\left(A\sin(\alpha) + B\sin(\beta)\right)\right]. \tag{4}$$

Then we convert the sum back to polar form as:

$$Ae^{j(\omega t+\alpha)} + Be^{j(\omega t+\beta)}$$

$$= e^{j\omega t} \cdot \sqrt{\left[A\cos(\alpha) + B\cos(\beta)\right]^2 + \left[A\sin(\alpha) + B\sin(\beta)\right]^2}$$

$$\bullet\, e^{j\cdot\tan^{-1}\left[\frac{A\sin(\alpha)+B\sin(\beta)}{A\cos(\alpha)+B\cos(\beta)}\right]}$$

$$= \sqrt{\left[A\cos(\alpha) + B\cos(\beta)\right]^2 + \left[A\sin(\alpha) + B\sin(\beta)\right]^2}$$

$$\bullet\, e^{j\left\{\omega t+\tan^{-1}\left[\frac{A\sin(\alpha)+B\sin(\beta)}{A\cos(\alpha)+B\cos(\beta)}\right]\right\}}. \tag{5}$$

(The "•" symbol in Eq. (5), needed for text wraparound reasons, simply means multiply.) So, Eq. (5) tells us: the sum of two equal-frequency complex exponentials is merely a scalar magnitude factor multiplied by a unity-amplitude complex exponential term. Next we use Eq. (5) to obtain the expressions for the sum of two real-valued sinusoids.

The Sum of Two Cosine Functions

The following shows the derivation of the cosine expressions in Table 1. Equating the real parts of both sides of Eq. (5) yields the desired (but messy) equation for the sum of two arbitrary equal-frequency cosine functions as:

$$A\cos(\omega t + \alpha) + B\cos(\omega t + \beta)$$

$$= \sqrt{\left[A\cos(\alpha) + B\cos(\beta)\right]^2 + \left[A\sin(\alpha) + B\sin(\beta)\right]^2}$$

$$\bullet\cos\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha) + B\sin(\beta)}{A\cos(\alpha) + B\cos(\beta)}\right]\right\}. \tag{6}$$

Eq. (6) is the general equation listed in the second row of Table 1. Substituting the different values for α, β, *B* from the left column of Table 1 into Eq. (6), plus recalling the following identities:

- $\cos^2(\theta) + \sin^2(\theta) = 1$
- sqrt{[1 + cos(θ)]/2} = cos(θ/2)
- tan(θ/2) = sin(θ)/[1 + cos(θ)],

allow us to obtain the expressions in the right column of the remaining rows in Table 1.

The Sum of Two Sine Functions

Equating the imaginary parts of both sides of Eq. (5) leads us to the desired equations for the sum of two general equal-frequency sine functions. As such, the expressions for the sum of two sine functions can be found by replacing all "cos(ω*t*..." terms in Table 1 with "sin(ω*t*..." to create Table 2.

The Sum of a Cosine Function and a Sine Function

We find the equation for the sum of a general cosine function and a general sine function, having the same frequencies, by recalling that sin(θ) = cos(θ − π/2) and using Eq. (6) as:

$$A\cos(\omega t + \alpha) + B\sin(\omega t + \beta) = A\cos(\omega t + \alpha) + B\cos(\omega t + \beta - \pi/2)$$

$$= \sqrt{[A\cos(\alpha) + B\cos(\beta - \pi/2)]^2 + [A\sin(\alpha) + B\sin(\beta - \pi/2)]^2}$$

$$\bullet \cos\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha) + B\sin(\beta - \pi/2)}{A\cos(\alpha) + B\cos(\beta - \pi/2)}\right]\right\}. \qquad (7)$$

Knowing that cos(θ − π/2) = sin(θ) and sin(θ − π/2) = −cos(θ), we modify Eq. (7) as:

$$A\cos(\omega t + \alpha) + B\sin(\omega t + \beta)$$

$$= \sqrt{[A\cos(\alpha) + B\sin(\beta)]^2 + [A\sin(\alpha) - B\cos(\beta)]^2}$$

$$\bullet \cos\left\{\omega t + \tan^{-1}\left[\frac{A\sin(\alpha) - B\cos(\beta)}{A\cos(\alpha) + B\sin(\beta)}\right]\right\}. \qquad (8)$$

Eq. (8) is the general equation listed in the second row of Table 3. Substituting the different values for α, β, *B* from the left column of Table 3 into Eq. (8), plus recalling the trig identities listed below Eq. (6), allow us to obtain the expressions in the right column of the remaining rows in Table 3.

References

[1] McClellan, J., Schafer, R., Yoder, M., *DSP First: A Multimedia Approach*, Prentice Hall; Upper Saddle River, New Jersey, 1998, pp. 48-50.

[2] Smith, J., "A Sum of Sinusoids at the Same Frequency is Another Sinusoid at that Frequency", http://ccrma.stanford.edu/~jos/filters/Sum_Sinusoids_Same_Frequency.html.

[3] Smith, J., "Why Sinusoids are Important", http://www-ccrma.stanford.edu/~jos/mdft/Why_Sinusoids_Important.html.

[4] Anderson, J., ***Digital Transmission Engineering, 2/E***, IEEE Press; Piscataway, New Jersey, 2005, pp. 288-289.